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On Domain Walls of $\mathcal{N} = 1$ Supersymmetric Yang-Mills in Four Dimension

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We study the BPS domain walls of supersymmetric Yang-Mills for arbitrary gauge group G . We describe the degeneracies of domain walls interpolating between arbitrary pairs of vacua. A recently proposed large N duality sheds light on various aspects of such domain walls. In particular, for the case of $G = SU(N)$ the domain walls correspond to wrapped D -branes giving rise to a 2+1 dimensional $U(k)$ gauge theory on the domain wall with a Chern-Simons term at level N . This leads to a counting of BPS degeneracies of domain walls in agreement with expected results.

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1. Introduction

$\mathcal{N} = 1$ Yang-Mills in $d = 4$ for a gauge group G admits $V = c_2(G)$ vacua where $c_2(G)$ denotes the dual Coxeter number of the group. In particular the $U(1)$ R-symmetry is anomalous, having a \mathbf{Z}_{2V} anomaly free subgroup, which is spontaneously broken to \mathbf{Z}_2 by the gaugino condensate

$$\langle \lambda\lambda \rangle = \Lambda^3 \omega$$

where $\omega^V = 1$. This raises the possibility of having domain walls. In fact one can consider BPS saturated domain walls since the value of the superpotential in each vacuum is proportional to the gaugino condensate. In particular the BPS tension of a domain wall connecting a pair of vacua which are separated by k units in phase of the gaugino condensate is given by

$$T = |\Delta W| \sim \Lambda^3 |1 - \omega^k|$$

The basic aim of this paper is to study the existence and degeneracy of domain walls which interpolate between any pair of such vacua, a question which clearly depends on k and the group G .

A related problem is the study of domain walls in $\mathcal{N} = 1$ supersymmetric Wess-Zumino model in 4 dimensions. The Wess-Zumino model with a superpotential W has as many vacua as the number of critical points of W , i.e. solutions to $dW = 0$ and we can consider domain walls interpolating between these vacua. This is equivalent to counting the kink solutions in its dimensional reduction to 2 dimensions (in which case it is often referred to as the $\mathcal{N} = 2$ Landau-Ginzburg model). In this context, the question of the number of BPS kinks (weighted with suitable signs [1]) has been completely solved [2] and is related to intersection theory in the context of the Picard-Lefschetz theory of singularities.

In fact the two problems are not unrelated: Consider the case of gauge group $G = SU(N)$. In this case, as noted in [2,3] the effective superpotential for the gaugino superfield, is identical to that of the supersymmetric sigma model on \mathbf{CP}^{N-1} , in terms of linear sigma model fields [4]. In particular both have N vacua. The number of BPS domain walls interpolating between vacua separated by k units for the \mathbf{CP}^{N-1} sigma model is known to be $N!/k!(N-k)!$ [5], and this suggests the same answer for the $SU(N)$ case. In fact this can be made even more plausible by noting that the compactification of the $SU(N)$ theory on the S^1 gives rise to a superpotential [6] which is exactly the same as the *mirror* description for the \mathbf{CP}^{N-1} discovered in [7]. Or if we compactify the 4d theory on T^2

the theory has a branch which corresponds to a sigma model on the moduli space of flat $SU(N)$ connections on T^2 , which is \mathbf{CP}^{N-1} [8,9].

In recent years there has been a fairly substantial amount of work focussing on BPS domain walls in the $\mathcal{N} = 1$ Yang-Mills theory from the viewpoint of low energy effective field theory [10]. A summary of many of the properties of the solutions can be found in [11]. One of the important features of the BPS domain walls is that they behave like D -branes for the QCD string as can be seen from the M theory formulation of $\mathcal{N} = 1$ Yang-Mills [12,13].

As for the degeneracy of the $SU(N)$ domain walls between vacua separated by one unit of the gaugino condensate phase, the embedding into M theory [12] resulted in finding a single domain wall [14]. This appears to be in contradiction with the expected answer of N , noted above, at least for the domain walls connecting adjacent vacua. We will be able to shed light on this apparent contradiction in this paper.

More recently, certain large N dualities for $\mathcal{N} = 1$ Yang-Mills were proposed [15,16,17]. In particular the duality proposed in [15] was reformulated in terms of M theory on G_2 holonomy manifolds in [18,19] and explained in terms of a geometric flop in [19].

We will use these recently discovered dualities to study the $\mathcal{N} = 1$ Yang-Mills domain walls. Our main insight will be to realise that the world-volume theory on the domain walls in these large N duals is a particular supersymmetric Chern-Simons-Yang-Mills theory. The enumeration of BPS domain walls is thus reduced to appropriately counting supersymmetric vacua of this theory. We will be able to recover the results anticipated based upon the equivalence with the supersymmetric \mathbf{CP}^{N-1} sigma model. Moreover we explain the apparent discrepancy with the result based on MQCD, by noting that the issue involves global boundary conditions on the wall - counting vacua in a toroidally compactified theory is not necessarily the same as counting vacua in Minkowski space.

The organization of this paper is as follows: In section 2 we discuss the prediction of the number of domain walls interpolating between adjacent vacua using the dual G_2 holonomy geometry, as well as from the Type IIA description involving RR-flux. In section 3 we use the Type IIA superstring description of the domain wall to compute the degeneracies of all the domain walls for arbitrary separation of vacua. In particular we show that the theory on the domain wall interpolating between vacua separated by k units, is an $\mathcal{N} = 1$ $U(k)$ gauge theory, with level N Chern-Simons term, with an extra scalar adjoint field. In section 4 we find the degeneracies of domain walls for arbitrary group G by considering the compactification of the theory to 3 and 2 dimensions and using the dual Landau-Ginzburg

description. This turns out to provide a very simple general answer for the degeneracies of BPS domain walls for all gauge groups, based on the Dynkin diagram and the Dynkin numbers of the gauge group. We also comment on how these general results might be derived in the context of G_2 holonomy geometries dual to $\mathcal{N} = 1$ systems.

2. Domain Walls of $\mathcal{N} = 1$ Yang-Mills

In the M theory formulation [18,19] of the duality of [15], one considers a G_2 holonomy geometry, which is topologically $(S^3 \times \mathbf{R}^4)/\mathbf{Z}_N$, where \mathbf{Z}_N acts differently in the IR versus the UV region, in a continuous way [19]. We will be mainly interested in the IR description where \mathbf{Z}_N acts on S^3 freely and produces a Lens space. From the Type IIA perspective, if we choose the eleventh dimension to be the fiber of the Hopf map $S^3/\mathbf{Z}_N \rightarrow S^2$ this corresponds to an S^2 geometry with N units of RR-flux through it. The domain wall corresponds to an $M5$ -brane wrapped over S^3/\mathbf{Z}_N , which in Type IIA theory is realized by a $D4$ -brane wrapped over S^2 (with N units of RR flux through it). Each such domain wall shifts the vacuum by one unit, as is evident from the identification of the supersymmetric vacua with the geometry [15].

Let us consider the theory on a single $M5$ -brane at low energies. This theory is free and in particular contains a 2-form field, b . If we wrap the $M5$ -brane on a manifold M , then there arises the possibility of turning on a topologically non-trivial, but *flat* b -field on M . Such choices are classified by $H^2(M, U(1))$ and these b -field backgrounds are the analogue of backgrounds with discrete torsion in string theory [20,21]. However $H^2(S^3/\mathbf{Z}_N, U(1))$ is trivial, so we do not have the freedom to turn on any discrete torsion. In other words there is a single domain wall in $3 + 1$ dimensions. Let us now consider compactifying one spatial direction on a circle. In this case we can turn on discrete torsion along the torsion class 2-cocycles formed from the 1-cocycle on the circle and the non-trivial 1-cocycles in $H^1(S^3/\mathbf{Z}_N, U(1))$ since,

$$H^2(S^3/\mathbf{Z}_N \times S^1) \cong \mathbf{Z}_N$$

Thus upon going to $2+1$ dimensions on a circle, we do get N BPS domain walls interpolating between adjacent vacua, as expected from considering the solitons of the \mathbf{CP}^{N-1} supersymmetric sigma model. If we consider the circle going down to 3 dimensions as the “11”-th direction, then from the perspective of Type IIA theory the domain wall corresponds to a $D4$ -brane wrapped over S^3/\mathbf{Z}_N and in that case we can turn on any of the N inequivalent flat connections on the brane world-volume [22]. These correspond to the N choices of discrete torsion in the M theory perspective.

2.1. Type IIA perspective in 3+1 dimensions

There is yet another way to understand this domain wall degeneracy, and that is from the perspective of Type IIA theory where instead we now consider the “11-th” direction to be the fibers of the Hopf map $S^3/\mathbf{Z}_N \rightarrow S^2$. The wrapped $M5$ -brane domain wall goes over to a $D4$ -brane wrapped around S^2 . The existence of the $U(1)$ gauge field on the wrapped D -brane is related to the fact that the fundamental string can end on the D -brane and thus provides a source for it. Since the domain wall is a wrapped D -brane and the QCD string in this context is identified with the fundamental string, this is consistent with the observation [12],[13] that the QCD string can end on the Yang-Mills domain wall.

On the world-volume of the domain wall we have a $2 + 1$ -dimensional $U(1)$ gauge theory with $\mathcal{N} = 1$ supersymmetry, together with a scalar multiplet, giving the normal position of the domain wall. This can be viewed as the multiplet we would have gotten if we did not have any flux through the S^2 , in which case we would have gotten an $\mathcal{N} = 2$ multiplet in $2 + 1$ dimensions on the domain wall. However the flux on the domain wall breaks this to $\mathcal{N} = 1$ in the following way: The gauge field picks up a Chern-Simons term of level N . This follows from the fact that on the $D4$ -brane world-volume there is the interaction

$$\int \tilde{A} \wedge F \wedge F = \int \tilde{F} \wedge A \wedge dA$$

where \tilde{A} is the bulk RR-gauge potential of Type IIA string theory and F denotes the field strength of the $U(1)$ gauge field A on the $D4$ -brane. Since $\int \tilde{F}$ over S^2 is N , this gives a Chern-Simons term of level N (after one takes into account the correct normalizations in the action). The existence of this interaction is consistent with the fact that without RR-flux, the Type IIA theory has twice the amount of supersymmetry, so the $D4$ -brane in the theory without flux would have $\mathcal{N} = 2$ supersymmetry on its world-volume. Turning on RR-flux breaks the bulk Type IIA supersymmetry by half and that induces the above Chern-Simons interaction on the $D4$ -brane, which breaks its supersymmetry by half also. Note that if we turn on a field strength F on the domain wall, say as a delta function, it serves as a source for the gauge field A , with N units of charge, due to this Chern-Simons term. This charge can be neutralized by N fundamental strings ending on the domain wall. This is to be identified with the Baryon vertex on the domain wall, and is consistent with the fact that a $D2$ -brane wrapped over S^2 can be viewed as the Baryon vertex (cf. [23], [24]) of the four dimensional gauge theory, which in turn is equivalent to turning on a delta function flux for F on the $D4$ -brane worldvolume.

The $U(1)$ gauge theory on the $2 + 1$ dimensional world-volume of the domain wall is massive, due to this interaction. This mass is identified with the only mass scale in the problem which is the physical scale of $\mathcal{N} = 1$ Yang-Mills. Note that the existence of the CS term and the generation of the mass for the gauge field implies that there are no long range propagating massless modes (other than the normal deformation of the domain wall) on the domain wall.

Even though the gauge theory on the domain wall is massive, the zero modes survive and lead to a topological counting of the number of vacua [25]. This has been studied mostly in compact spaces, in particular when we compactify the theory on the wall from $2 + 1$ to $0 + 1$ on a T^2 in which case we get N vacua, i.e. the number of conformal blocks of RCFT, for the $U(1)$ theory at level N . Note, that this corresponds to the statement that in the four dimensional super Yang-Mills theory compactified on T^2 , there are N domain walls between neighbouring vacua.

Even if we consider the theory on S^1 instead of the torus, the above results suggest that we should still obtain N vacua for the $U(1)$ Chern-Simons theory at level N . We also note that since this $U(1)$ theory is free, the uncompactified theory has one bosonic zero energy state, consistent with the M theory description above.

We have thus considered the number of domain walls between adjacent vacua in supersymmetric $SU(N)$ gauge theory from various points of view and also resolved the puzzle of their degeneracies in connection with the M theory realization [14]. It is natural to ask what happens when we consider domain walls separating vacua which are not adjacent. It turns out that the last viewpoint, namely that of Type IIA in $3+1$ dimensions with k D4 branes wrapping the S^2 is the most fruitful for this question and we consider that next. However, we will make comments about the description involving M theory, or Type IIA, on G_2 -holonomy geometries at the end of section 4, after we describe the expected result for domain wall degeneracies for arbitrary gauge groups.

3. Gauge Theory on the Domain Wall

Consider the Type IIA background considered in [15]. The spacetime contains an S^2 with N units of RR-flux through it. Consider k D4 branes wrapping this S^2 . These k D4-branes are the domain wall which interpolates between vacuum l and $l + k \pmod{N}$ of the $\mathcal{N} = 1$ theory.

The discussion in the previous section easily extends to the case of more than one brane and we obtain a $U(k)$ gauge theory on the $2 + 1$ dimensional world-volume of the domain wall with the same matter content as an $\mathcal{N} = 2$ supersymmetric theory (which has 4 supercharges) but broken to an $\mathcal{N} = 1$ supersymmetric theory, by a Chern-Simons term for the gauge field at level N . We can view this as an $\mathcal{N} = 1$ supersymmetric $U(k)$ gauge theory with Chern-Simons coupling of level N coupled to a massless scalar multiplet in the adjoint representation. We would like to compute the number of ground states of this theory. To be more precise we compute the number of ground states of this theory compactified on T^2 , and identify it with the number of BPS domain walls interpolating between vacua which are k units apart. To be more precise we will compute the net number of vacua of the theory weighted by $(-1)^F$, except for the overall $U(1)$ factor which gives for every ground state an opposite statistic ground state due to the fermionic zero mode in the matter multiplet. This is because the Witten index of the full $U(k)$ theory is zero, since the “central $U(1)$ -theory” is free. This $U(1)$ corresponds to the usual center of mass motion of the domain wall which gives it the correct spacetime supersymmetry multiplet structure. If the index of the full $U(k)$ theory were non-zero the domain wall would not be in a supermultiplet. This is also related to computing $\text{Tr}(-1)^F F$ [1] for each kink sector.

In other words we regard the $\mathcal{N} = 1$ $U(k)$ gauge theory coupled to the adjoint scalar, as being a

$$U(k) = \frac{U(1) \times SU(k)}{\mathbf{Z}_k}$$

gauge theory, and we will be interested in computing the Witten index of this system, modulo the trivial doubling (leading to zero) coming from the fermionic zero mode in the $U(1)$ factor.

Thus we have to find the number of ground states of the $U(1)$ theory at level N times that of the $\mathcal{N} = 1$ $SU(k)$ system at level N coupled to an adjoint scalar. The number of ground states coming from the $U(1)$ system (ignoring the fermion zero mode in the matter adjoint) is N . For the $SU(k)$ system, we have moduli given by the vev of the adjoint scalar $\langle \phi \rangle$. If we compactify the theory on a T^2 we have to integrate over all allowed moduli. For any non-vanishing vev of the adjoint scalar the $SU(k)$ gauge symmetry is broken to a factor, which includes at least one $U(1)$ given by the direction of the vev in the adjoint of $SU(k)$. Thus the corresponding contribution to Witten index vanishes from such points due to the extra fermionic zero mode in the matter system along that $U(1)$ direction. Thus the only point in moduli space which can contribute to the index is the origin.

Consider adding a mass term for the $SU(k)$ components of ϕ (this can be done, for example by making the gauge theory an $\mathcal{N} = 2$ system). This changes the behavior of the system for large vevs of ϕ without affecting the theory near the origin of ϕ . This could have potentially changed the Witten index, because we are changing the behavior of the system at infinity in the field space. However, as we already argued, away from the origin there is no contribution to Witten index, and so with this deformation we would not be modifying the Witten index computation.

Thus we need to compute the Witten index for an $\mathcal{N} = 2$ $SU(k)$ gauge theory at level N . This has been done in [26][27] following the work of [28], and it is given by the number of level $N - k$ representations of affine $SU(k)$ theory. The answer is

$$I_{SU(k)} = Tr_{SU(k)}(-1)^F = \frac{(N-1)!}{(k-1)!(N-k)!}$$

To find the total number of vacua we include the contribution of the $U(1)$ piece, which gives a factor of N ; taking the \mathbf{Z}_k quotient which takes us from $U(1) \times SU(k)$ to the $U(k)$ theory, gives an additional factor of $1/k$ and we obtain

$$I'_{U(k)} = \frac{N!}{k!(N-k)!}$$

(where the prime denotes deleting the zero mode from the $U(1)$ fermionic factor). This answer is exactly as expected for the net number of BPS domain walls of the $\mathcal{N} = 1$ $SU(N)$ gauge theory interpolating between vacua k units apart.

Note that the above answer applies to the four dimensional theory compactified on T^2 . The number of walls in the four dimensional theory in non-compact \mathbf{R}^4 is given by the index of the three dimensional Chern-Simons theory on \mathbf{R}^3 which may not be the same as the index on $T^2 \times \mathbf{R}$, as we have seen already for the case of $k = 1$.

4. The BPS Domain Walls for $d = 4, \mathcal{N} = 1$ Supersymmetric Gauge Theory for Arbitrary Gauge Group

In this section we expand upon the known results in the literature and compute the (net number of) BPS domain walls for $d = 4, \mathcal{N} = 1$ Supersymmetric Gauge theories for arbitrary gauge group compactified on a circle or T^2 to 3 or 2 dimensions. Before we explain the derivation, we state the result which is surprisingly simple: For each group G consider the affine Dynkin diagram. Associate to each node a fermion with a “ $U(1)$ charge” given

by the corresponding Dynkin index. Note that the sum of the Dynkin indices is $c_2(G)$, i.e., the total number of vacua of the $\mathcal{N}=1$ theory. Consider all the possible monomials of the fermions with a total “ $U(1)$ charge” of k . Then the net number of domain walls between vacua separated by k units (for $1 < k < c_2(G)$) is given by the net number of fermions with total charge k , where the net number is counted relative to the $(-1)^F$. Note that for the $SU(N)$ case, the affine Dynkin diagram has N nodes each with Dynkin index 1, and this gives the expected answer $(-1)^k N!/k!(N-k)!$ for the number of domain walls separating vacua which are k units apart.

Upon compactifying $\mathcal{N}=1$ supersymmetric Yang-Mills theory on S^1 or T^2 one obtains an effective Landau-Ginzburg theory described as follows: Consider the affine Dynkin diagram of the group G , and for each node introduce a chiral field Y_i , whose imaginary part is periodic with period 2π . Consider a theory with 4 supercharges given by a holomorphic superpotential

$$W = \sum_i e^{-Y_i}$$

with the constraint that $\sum_i a_i Y_i = 0$, where a_i are the associated Dynkin indices. It is a simple exercise to show that this Landau-Ginzburg theory has $c_2(G) = \sum a_i$ vacua, and that the value of the superpotential is given by $W = r\omega$ for some real parameter r , where ω is a $c_2(G)$ -th root of unity.

For the compactification on a circle, this superpotential was derived in [6][29] by embedding the system into F -theory, and from a field theory analysis in [30][31]. In compactification to 2 dimensions this has been argued to arise by noting that the compactification of the gauge theory on T^2 leads to an $\mathcal{N}=2$ supersymmetric sigma model on the moduli space of flat G -connections on T^2 which in turn is given by a weighted projective space WP^{a_i} where the weights are given by the Dynkin indices of the corresponding affine Dynkin diagram [8,9]. This corresponds to a $U(1)$ linear sigma model with $\mathcal{N} = (2,2)$ in $d = 2$ with $\text{rank}(G) + 1$ fields of charges given by the indices a_i , as shown in figure 1 (which is borrowed from [9]). Then, as discussed in [7] mirror symmetry maps this sigma model to a Landau-Ginzburg theory with the superpotential given above.

The computation of the BPS domain wall solutions thus reduces to the computation of the intersection number of vanishing cycles of the corresponding Landau-Ginzburg theory, as discussed in detail in [2]. In fact it is more convenient to compute this intersection number by using the mirror symmetry to the weighted projective space, as was done in [5]. In this case one considers a certain intersection number of D -branes in the sigma model,